

Isospectral Mathieu-Hill Operators

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Abstract

In this paper we prove that the spectra of the Mathieu-Hill Operators with potentials $ae^{-i2\pi x} + be^{i2\pi x}$ and $ce^{-i2\pi x} + de^{i2\pi x}$ are the same if and only if $ab = cd$, where a, b, c and d are complex numbers. This implies some corollaries about the extension of Harrell-Avron-Simon formula. Moreover, we find explicit formulas for the eigenvalues and eigenfunctions of the t -periodic boundary value problem for the Hill operator with Gasymov's potential.

Key Words: Mathieu-Hill operator, Spectrum, Isospectral operators.

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Let $H(a, b)$ be the Hill operator generated in $L_2(-\infty, \infty)$ by the expression

$$-y''(x) + q(x)y(x) \quad (1)$$

with potential

$$q(x) = ae^{-i2\pi x} + be^{i2\pi x}, \quad (2)$$

where a and b are complex numbers. It is well-known that (see [4, 8]) the spectrum $S(H(a, b))$ of the operator $H(a, b)$ is the union of the spectra $S(H_t(a, b))$ of the operators $H_t(a, b)$ for $t \in (-\pi, \pi]$, where $H_t(a, b)$ is the operator generated in $L_2[0, 1]$ by (1) with potential (2) and by the boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0). \quad (3)$$

First we prove that if $ab = cd$, then

$$S(H(a, b)) = S(H(c, d)), \quad S(H_t(a, b)) = S(H_t(c, d)) \quad (4)$$

for all $t \in (-\pi, \pi]$. For this we obtain the asymptotic formulas, uniform with respect to $t \in [\rho, \pi - \rho]$, for eigenvalues and eigenfunctions of the operators H_t , where ρ is a fixed number from the interval $(0, \frac{\pi}{2})$. Note that the formula $f(k, t) = O(h(k))$ is said to be uniform with respect to t in a set I if there exist positive constants M and N , independent of t , such that $|f(k, t)| < M |h(k)|$ for all $t \in I$ and $|k| \geq N$.

To obtain the uniform asymptotic formulas for eigenvalues $\lambda_n(t)$ and corresponding normalized eigenfunctions $\Psi_{n,t}(x)$ for $t \in [\rho, \pi - \rho]$, as $n \rightarrow \infty$ we use the formulas

$$(\lambda_n(t) - (2\pi n + t)^2)(\Psi_{n,t}, e^{i(2\pi n + t)x}) = (q\Psi_{n,t}, e^{i(2\pi n + t)x}) \quad (5)$$

and

$$(\lambda_n(t) - (2\pi(n - k) + t)^2)(\Psi_{n,t}, e^{i(2\pi(n - k) + t)x}) = (q\Psi_{n,t}, e^{i(2\pi(n - k) + t)x}), \quad (6)$$

where (\cdot, \cdot) is the inner product in $L_2[0, 1]$. Formulas (5) and (6) can be obtained from

$$-\Psi''_{n,t}(x) + q(x)\Psi_{n,t} = \lambda_n(t)\Psi_{n,t}(x) \quad (7)$$

by multiplying $e^{i(2\pi n+t)x}$ and $e^{i(2\pi(n-k)+t)x}$ respectively.

The uniform asymptotic formulas for the operator $L_t(q)$, with $q \in L_1[0, 1]$, generated in $L_2[0, 1]$ by (1) and (3) is obtained in [9], where we proved the following:

The large eigenvalue $\lambda_n(t)$ and the corresponding eigenfunction $\Psi_{n,t}(x)$ of the operator $L_t(q)$ for $t \neq 0, \pi$, satisfy the following asymptotic formulas

$$\lambda_n(t) = (2\pi n + t)^2 + O\left(\frac{\ln |n|}{n}\right), \quad \Psi_{n,t}(x) = e^{i(2\pi n+t)x} + O\left(\frac{1}{n}\right). \quad (8)$$

These asymptotic formulas are uniform with respect to t in $[\rho, \pi - \rho]$, where ρ is a fixed number from $(0, \frac{\pi}{2})$. There exists a positive number $N(\rho)$, independent of t , such that the eigenvalues $\lambda_n(t)$ for $t \in [\rho, \pi - \rho]$ and $|n| > N(\rho)$ are simple.

In [9], we obtained (8) by iteration of the formula (5). However, for the convenience of the readers and taking into account that we need to consider the terms of the asymptotic formulas in detail, we repeat the iteration here. Using (2) in (5) we get

$$(\lambda_n(t) - (2\pi n + t)^2)(\Psi_{n,t}(x), e^{i(2\pi n+t)x}) = \sum_{n_1} q_{n_1}(\Psi_{n,t}(x), e^{i(2\pi(n-n_1)+t)x}), \quad (9)$$

where

$$q_n = (q(x), e^{i2\pi nx}), \quad q_{-1} = a, \quad q_1 = b, \quad q_n = 0, \quad \forall n \neq \pm 1. \quad (10)$$

In (6) replacing k by n_1 and then using (2) we get

$$(\lambda_n(t) - (2\pi(n - n_1) + t)^2)(\Psi_{n,t}, e^{i(2\pi(n-n_1)+t)x}) = \sum_{n_2} q_{n_2}(\Psi_{n,t}, e^{i(2\pi(n-n_1-n_2)+t)x}). \quad (11)$$

Let

$$U(n, t) = \{\lambda \in \mathbb{C} : |\lambda - (2\pi n + t)^2| \leq 1\}, \quad n \in \mathbb{Z}.$$

By (8) the disk $U(n, t)$ for $t \in [\rho, \pi - \rho]$ and $|n| > N(\rho)$ contains only one simple eigenvalue denoted by $\lambda_n(t)$. Moreover, it is well known that [4] for $t = 0$ and $|n| \gg 1$ the disk $U(n, t)$ contains two eigenvalues, denoted here by $\lambda_n(0)$ and $\lambda_{-n}(0)$. One can readily see that if $t \in [\rho, \pi - \rho]$, $k \neq 0$ and $|n| > N(\rho) \gg 1$, then

$$|\lambda_n(t) - (2\pi(n - k) + t)^2| > |n| \rho, \quad |\lambda - (2\pi(n - k) + t)^2| > |n| \rho. \quad (12)$$

for all $t \in [\rho, \pi - \rho]$ and $\lambda \in U(n, t)$.

Now we iterate (9) as follows. By (12), the last multiplicand $(\Psi_{n,t}, e^{i(2\pi(n-n_1)+t)x})$ in (9) can be replaced with the right-hand side of (11) divided by $\lambda_n(t) - (2\pi(n - n_1) + t)^2$. Doing this replacement we get

$$(\lambda_n(t) - (2\pi n + t)^2)(\Psi_{n,t}, e^{i(2\pi n+t)x}) = \sum_{n_1, n_2} \frac{q_{n_1} q_{n_2}(\Psi_{n,t}, e^{i(2\pi(n-n_1-n_2)+t)x})}{\lambda_n(t) - (2\pi(n - n_1) + t)^2}. \quad (13)$$

Now we isolate the terms in right-hand side of (13) containing the multiplicand

$(\Psi_{n,t}, e^{i(2\pi n+t)x})$ which occurs in the case $n_1 + n_2 = 0$ and apply the above replacement

to the other terms (i.e., case $n_1 + n_2 \neq 0$) to get

$$(\lambda_n(t) - (2\pi n + t)^2)(\Psi_{n,t}, e^{i(2\pi n + t)x}) = \sum_{n_1} \frac{q_{n_1} q_{-n_1}(\Psi_{n,t}, e^{i(2\pi n + t)x})}{\lambda_n(t) - (2\pi(n - n_1) + t)^2} +$$

$$\sum_{n_1, n_2, n_3} \frac{q_{n_1} q_{n_2} q_{n_3}(\Psi_{n,t}, e^{i(2\pi(n - n_1 - n_2 - n_3) + t)x})}{(\lambda_n(t) - (2\pi(n - n_1) + t)^2)(\lambda_n(t) - (2\pi(n - n_1 - n_2) + t)^2)}. \quad (14)$$

Repeating this process m -times (i.e., in the second row of (14), isolating the terms containing the multiplicand $(\Psi_{n,t}, e^{i(2\pi n + t)x})$ applying the above replacement to the other terms, etc.) we obtain

$$(\lambda_n(t) - (2\pi n + t)^2)(\Psi_{n,t}, e^{i(2\pi n + t)x}) = A_m(\lambda_n(t))(\Psi_{n,t}, e^{i(2\pi n + t)x}) + R_{m+1}(\lambda_n(t)), \quad (15)$$

where $A_m(\lambda) = \sum_{k=1}^m a_k(\lambda)$,

$$a_k(\lambda) = \sum_{n_1, n_2, \dots, n_k} \frac{q_{n_1} q_{n_2} \dots q_{n_k} q_{-n_1 - n_2 - \dots - n_k}}{\prod_{s=1, 2, \dots, k} [\lambda - (2\pi(n - n_1 - n_2 - \dots - n_s) + t)^2]}, \quad (16)$$

$$R_{m+1}(\lambda) = \sum_{n_1, n_2, \dots, n_{m+1}} \frac{q_{n_1} q_{n_2} \dots q_{n_m} q_{n_{m+1}}(\Psi_{n,t}(x), e^{i(2\pi(n - n_1 - n_2 - \dots - n_{m+1}) + t)x})}{\prod_{s=1, 2, \dots, m+1} [\lambda - (2\pi(n - n_1 - n_2 - \dots - n_s) + t)^2]} \quad (17)$$

and by (10)

$$\{n_1, n_2, \dots, n_k, -n_1 - n_2 - \dots - n_k\} \subset \{-1, 1\}. \quad (18)$$

Let k be an even number: $k = 2p$. Then $-n_1 - n_2 - \dots - n_k$ is also an even number, since the numbers n_1, n_2, \dots, n_k are -1 or 1 (see (18)). Therefore by (10), $q_{-n_1 - n_2 - \dots - n_k} = 0$ and by (16)

$$a_{2p}(\lambda, t) = 0, \quad \forall p = 1, 2, \dots \quad (19)$$

Let k be an odd number: $k = 2p - 1$. Then the set in the left-hand side of (18) contains $2p$ numbers that are -1 or 1 and their total sum is 0 . Hence p of those numbers are -1 and p of those are 1 . Therefore, by (10) and (16), $a_k(\lambda, t)$ for $k = 2p - 1$ is the sum of 2^k terms of the form

$$(ab)^p \prod_{s=1, 2, \dots, k} (\lambda - (2\pi(n - n_1 - n_2 - \dots - n_s) + t)^2)^{-1}. \quad (20)$$

Thus we have

$$a_{2p-1}(\lambda, t) = 2^{2p-1} (ab)^p f_p(\lambda, t), \quad (21)$$

where $f_p(\lambda, t)$ does not depend on a and b and, by (12), satisfies the following, uniform with respect to $t \in [\rho, \pi - \rho]$, equality

$$f_p(\lambda, t) = O((n\rho)^{-2p+1}) \quad (22)$$

for $\lambda \in U(n, t)$. In the same way we obtain

$$R_{m+1} = O((2M)^{m+1} (n\rho)^{-m-1}), \quad (23)$$

where $M = \max\{|a|, |b|\}$. Letting m tend to infinity in (15) and using (23) we obtain that $\lambda_n(t)$ is a root of the equation

$$\lambda - (2\pi n + t)^2 = A(\lambda, t, ab), \quad (24)$$

where, by (19),

$$A(\lambda, t, ab) = \sum_{p=1}^{\infty} a_{2p-1}(\lambda, t). \quad (25)$$

It follows from (21) and (22) that $A(\lambda, t, ab)$, for fixed t , is an analytic function of $\lambda \in U(n, t)$ satisfying the following, uniform with respect to $t \in [\rho, \pi - \rho]$, asymptotic formula

$$A(\lambda, t, ab) = O(n^{-1}).$$

Therefore the inequality

$$|A(\lambda, t, ab)| < |\lambda - (2\pi n + t)^2| \quad (26)$$

holds for all λ from the boundary of $U(n, t)$. Since the function $\lambda - (2\pi n + t)^2$ has one root in the set $U(n, t)$, by the Rouché's theorem (24) has also one root in the same set. On the other hand, $\lambda_n(t)$ is a root of (24) lying in $U(n, t)$. Therefore $\lambda \in U(n, t)$ is an eigenvalue of $H_t(a, b)$ if and only if it is a root of (24).

Theorem 1 *If $ab = cd$, then (4) holds.*

Proof. One can readily see from (21) and (25) that if $ab = cd$ then

$$A(\lambda, t, ab) = A(\lambda, t, cd). \quad (27)$$

Let $\mu_n(t)$ be the eigenvalue of $H_t(c, d)$ lying in $U(n, t)$. By (27), both $\lambda_n(t)$ and $\mu_n(t)$ are the roots of the same equation (24) which has unique root in $U(n, t)$. Therefore we have

$$\lambda_n(t) = \mu_n(t), \quad \forall t \in [\rho, \pi - \rho]. \quad (28)$$

On the other hand, $\lambda_n(t)$ and $\mu_n(t)$ are the roots of the equations

$$F(\lambda) = \cos t, \quad G(\lambda) = \cos t, \quad (29)$$

where $F(\lambda)$ and $G(\lambda)$ are the Hill's discriminants of the operators $H_t(a, b)$ and $H_t(c, d)$ respectively (see [6,9]). Since the eigenvalue $\lambda_n(t)$ for $t \in [\rho, \pi - \rho]$ and $|n| > N(\rho)$ is simple the set $\{\lambda_n(t) : t \in [\rho, \pi - \rho]\}$ is an analytic arc. By (28) and (29) the entire functions $F(\lambda)$ and $G(\lambda)$ coincide on this arc. Therefore these functions are identically equal in the complex plane and hence the eigenvalues of $H_t(a, b)$ and $H_t(c, d)$ are the roots of the same equation (29) for all $t \in (-\pi, \pi]$, that is, (4) holds ■

Remark 1 *Note that to prove Theorem 1 we investigated the simplest case $t \in [\rho, \pi - \rho]$. In the paper [10] we obtained the uniform asymptotic formulas in the more complicated case $t \in [0, \rho] \cup [\pi - \rho, \pi]$. In the same way one can prove Theorem 1 by using the formulas of [10]. Indeed, in [10] we proved that (see Theorem 2 and 4 of [10]) the eigenvalue $\lambda_n(t)$ for $t \in [0, \rho]$ and $n > N \gg 1$, is simple and satisfies the equality*

$$(\lambda - (2\pi n + t)^2 - A(\lambda, t))(\lambda - (2\pi n - t)^2 - A'(\lambda, t)) = B(\lambda, t)B'(\lambda, t), \quad (30)$$

where $A(\lambda, t)$ and $A'(\lambda, t)$ are defined as (25) and hence depend only on ab (see (8)-(14) of [10]). The functions $B(\lambda, t)$ and $B'(\lambda, t)$ are the sum of $b_{2n+2m-1}(\lambda, t)$ and $b'_{2n+2m-1}(\lambda, t)$ respectively for $m = 0, 1, 2, \dots$. Moreover, from (46), and (10) of [10] one can readily see that $2n + m$ of the indices $n_1, n_2, \dots, n_{2n+2m-1}, 2n - n_1 - n_2 - \dots - n_{2n+2m-1}$ taking part in $b_{2n+2m-1}(\lambda, t)$ are 1 and m indices of them are -1. This implies that the expression

$(b_{2n-1}(\lambda, t))^{-1}b_{2n+2m-1}(\lambda, t)$ depends only on ab , since

$$b_{2n-1}(\lambda, t) = b^{2n} \prod_{s=1}^{2n-1} (\lambda - (2\pi(n-s) + t)^2)^{-1}. \quad (31)$$

Similarly, the expression $(b'_{2n-1}(\lambda, t))^{-1}b'_{2n+2m-1}(\lambda, t)$ depends only on ab , where

$$b'_{2n-1}(\lambda, t) = a^{2n} \prod_{s=1}^{2n-1} (\lambda - (2\pi(n-s) - t)^2)^{-1}. \quad (32)$$

Therefore the right-hand side of (30) also depends only on ab , that is,

$$B(\lambda, t)B'(\lambda, t) = f(\lambda, t, ab) \quad (33)$$

Using these and arguing as in the proof of Theorem 1 we get the other proof of Theorem 1.

Theorem 1 shows that if all the eigenvalues of $H_t(a, b)$ for all values of $t \in (-\pi, \pi]$ are given then one can determine only ab . However, the following simple theorem shows that one can determine ab by given subsequence of the eigenvalues of $H_t(a, b)$ for some value of t .

Theorem 2 *If for some value of $t \in [0, \pi]$ and for some sequence $\{n_k\}$ the eigenvalues $\lambda_{n_k}(t, a, b) =: \lambda_{n_k}(t)$ of $H_t(a, b)$ are given, then one can constructively determine ab .*

Proof. Let $t \in (0, \pi)$. Then there exists $\rho \in (0, \frac{\pi}{2})$ such that $t \in [\rho, \pi - \rho]$. Without loss of generality and for simplicity of notation assume that $\lambda_n(t)$ for $n \geq N(\rho)$ are given. It follows from (24), (25), (21) and (22) that

$$\lambda_n(t) = (2\pi n + t)^2 + O\left(\frac{1}{n}\right).$$

Using it in (16) for $k = 1$ we obtain

$$\begin{aligned} a_1(\lambda_n(t)) &= \frac{ab}{(2\pi n + t)^2 + O(n^{-1}) - (2\pi(n-1) + t)^2} + \\ &\quad \frac{ab}{(2\pi n + t)^2 + O(n^{-1}) - (2\pi(n+1) + t)^2} = \\ &= \frac{ab}{2\pi(2\pi(2n-1) + 2t)} - \frac{ab}{2\pi(2\pi(2n+1) + 2t)} + O\left(\frac{1}{n^3}\right) = \frac{ab}{2(2\pi n + t)^2} + O\left(\frac{1}{n^3}\right). \end{aligned}$$

This with (24), (25), (21) and (22) implies that

$$\lambda_n(t) = (2\pi n + t)^2 + \frac{ab}{2(2\pi n + t)^2} + O\left(\frac{1}{n^3}\right).$$

From this we find ab by calculating the limit: $\lim_{n \rightarrow \infty} (\lambda_n(t) - (2\pi n + t)^2)2(2\pi n + t)^2$. If $t = 0$, then using the well-known [4] asymptotic formula

$$\lambda_n(t) = (2\pi n)^2 + \frac{ab}{2(2\pi n)^2} + O\left(\frac{1}{n^3}\right)$$

in the same way we determine ab . The case $t = \pi$ can be considered in the same way. ■

Theorem 3 *The following conditions are equivalent*

- 1) $ab = cd$
- 2) $S(H_t(a, b)) = S(H_t(c, d))$ for all $t \in (-\pi, \pi]$
- 3) $S(H_t(a, b)) = S(H_t(c, d))$ for some $t \in (-\pi, \pi]$
- 4) $\lambda_{n_k}(t, a, b) = \lambda_{n_k}(t, c, d)$ for some $t \in (-\pi, \pi]$ and for some sequence $\{n_k\}$.
- 5) $S(H(a, b)) = S(H(c, d))$.

Proof. By Theorem 1, 1) implies 2) and 5). It is clear that $2) \implies 3) \implies 4)$. By Theorem 2, $4) \implies 1)$. Thus 1), 2), 3) and 4) are equivalent. It remains to show that 5) implies at least one of them. By Theorem 5 of [10] there exists N such that for $|n| > N$ the component Γ_n of the spectrum of the operator H is separated simple analytic arc with end points $\lambda_n(0)$ and $\lambda_n(\pi)$. Therefore 5) implies 4). ■

Now we obtain some consequences of Theorem 1. First consequence is the generalization of formula (3.25) of [1], which is the extension of the asymptotic formula of Harrell-Avron-Simon [2, 7], for the case $a \neq b$. For simplicity of reading we write this result in notation of [1]. Let $\lambda_n^\pm(a, b)$ be the eigenvalue of the Mathieu operator with potential

$$ae^{-i2x} + be^{i2x} \quad (34)$$

and with periodic (if n is even) or antiperiodic (if n is odd) boundary condition. It is proved in [1] that if $b = a$, then the following asymptotic formula holds

$$\lambda_n^+ - \lambda_n^- = \pm 8a^n 4^{-n} ((n-1)!)^{-2} \left(1 - \frac{a^2}{4n^3} + O\left(\frac{1}{n^4}\right)\right). \quad (35)$$

It is clear that Theorem 1 continues to hold for (34), since this case can be reduced to the case (2) by substitution $s = \pi x$.

Corollary 1 *For every complex numbers a and b the following formula holds*

$$\lambda_n^+(a, b) - \lambda_n^-(ab) = \pm 8(ab)^{\frac{n}{2}} 4^{-n} ((n-1)!)^{-2} \left(1 - \frac{ab}{4n^3} + O\left(\frac{1}{n^4}\right)\right). \quad (36)$$

Proof. First proof of (36): Let $c = (ab)^{\frac{1}{2}}$. By Theorem 1, $\lambda_n^\pm(a, b) = \lambda_n^\pm(c, c)$. Therefore in (35) replacing a^2 with ab we obtain (36).

Second proof of (36): $\lambda_n^\pm(a, b) = n^2 + z$ for large n is an eigenvalue of periodic or antiperiodic boundary condition if and only if z is a root of the equation (18) of [3]:

$$(z - a(n, z))^2 = B^+(n, z)B^-(n, z),$$

where $a(n, z), B^+(n, z), B^-(n, z)$ are defined as $A(\lambda, t), B(\lambda, t), B'(\lambda, t)$ for $t = 0$. Hence by Remark 1 $a(n, z)$ and $B^+(n, z)B^-(n, z)$ depend only on ab . Therefore arguing as above we obtain (36) from (35). ■

Now we consider the operator $L_t(q)$ generated by (1) and (3) when

$$q \in L_1[0, 1], \quad q_n = 0, \forall n = 0, -1, -2, \dots \quad (37)$$

Spectral theory of the operator $L(q)$ generated in $L_2(-\infty, \infty)$ by the expression (1) with the potential q satisfying (37) and the additional condition $\sum_n |q_n| < \infty$ is studied by Gasymov [5].

Theorem 4 *(a) The eigenvalues of the operators $L_t(q)$ for $t \in (-\pi, \pi]$ with potential (37) are $(2\pi n + t)^2$, where $n \in \mathbb{Z}$. These eigenvalues for $t \neq 0, \pi$ are simple. The eigenvalues $(2\pi n)^2$ for $n \in \mathbb{Z} \setminus \{0\}$ and $(2\pi n + \pi)^2$ for $n \in \mathbb{Z}$ are double eigenvalues of $L_0(q)$ and $L_\pi(q)$*

respectively. The theorem continues to hold if (37) is replaced by

$$q \in L_1[0, 1], \quad q_n = 0, \forall n = 0, 1, 2, \dots \quad (38)$$

(b) Let $\Psi_{n,t}(x)$ be the eigenfunction of the operator $L_t(q)$ corresponding to the eigenvalue $(2\pi n + t)^2$ and normalized as

$$(\Psi_{n,t}, e^{i(2\pi n + t)x}) = 1, \quad (39)$$

where $t \neq 0, \pi$ and q satisfies (37). Then

$$\Psi_{n,t}(x) = e^{i(2\pi n + t)x} + \sum_{p \in \mathbb{N}} c_p(t) e^{i(2\pi(n+p) + t)x}, \quad (40)$$

where $c_1(t) = q_1 d_1(t)$, $d_p(t) = ((2\pi n + t)^2 - (2\pi(n+p) + t)^2)^{-1}$,

$$c_2(t) = d_2(t) \left(q_2 + \frac{q_1 q_1}{(2\pi n + t)^2 - (2\pi(n+1) + t)^2} \right),$$

$$c_p = d_p \left(q_p + \sum_{k=1}^{p-1} \sum_{n_1, n_2, \dots, n_k} \frac{q_{n_1} q_{n_2} \dots q_{n_k} q_{p-n_1-n_2-\dots-n_k}}{\prod_{s=1}^k [(2\pi n + t)^2 - (2\pi(n + p - n_1 - n_2 - \dots - n_s) + t)^2]} \right) \quad (41)$$

for $p = 3, 4, \dots$ and

$$\{n_1, n_2, \dots, n_s, p - n_1 - n_2 - \dots - n_s\} \subset \mathbb{N} =: \{1, 2, \dots\} \quad (42)$$

for $s = 1, 2, \dots, p-1$. The theorem continues to hold if (37) is replaced by (38) and \mathbb{N} in formulas (40) and (42) is replaced by $-\mathbb{N}$.

Proof. (a) Formulas (15)-(17) for $q \in L_1[0, 1]$ with estimations that guaranties the equality $\lim_{m \rightarrow \infty} R_{m+1} = 0$ are proved in [9]. Since at least one of the indices $n_1, n_2, \dots, n_k, -n_1 - n_2 - \dots - n_k$ is not positive number, by (16) and (37), $a_k(\lambda_n(t)) = 0$ and hence $A_m(\lambda_n(t)) = 0$. Therefore letting m tend to infinity and using (8) and (15) we obtain $\lambda_n(t) = (2\pi n + t)^2$ for $t \in [\rho, \pi - \rho]$ and $n \gg 1$. Now arguing as in the proof of Theorem 1 we get the proof of this theorem in case (37). Case (38) can be proved in the same way.

(b) Let $\Psi_{n,t}(x)$ be the normalized eigenfunction of the operator $L_t(q)$ corresponding to the eigenvalue $(2\pi n + t)^2$ and $t \neq 0, \pi$. (In the end we prove that there exists an eigenfunction of the operator $L_t(q)$ satisfying (39). For simplicity of notation we denote it also by $\Psi_{n,t}$.) Since $\{e^{i(2\pi(n+p) + t)x} : p \in \mathbb{Z}\}$ is an orthonormal basis we have

$$\Psi_{n,t}(x) - (\Psi_{n,t}, e^{i(2\pi n + t)x}) e^{i(2\pi n + t)x} = \sum_{p \in \mathbb{Z} \setminus \{0\}} (\Psi_{n,t}, e^{i(2\pi(n+p) + t)x}) e^{i(2\pi(n+p) + t)x}. \quad (43)$$

To find $(\Psi_{n,t}, e^{i(2\pi(n+p) + t)x})$ we iterate (6) (in (6) replace $-k$ with p) by using

$$(q \Psi_{n,t}, e^{i(2\pi(n+p) + t)x}) = \sum_{n_1} q_{n_1} (\Psi_{n,t}, e^{i(2\pi(n+p-n_1) + t)x}) \quad (44)$$

(see (14) of [9]) and the equality $\lambda_n(t) = (2\pi n + t)^2$ (see (a)). Namely, (6) with (44) implies

$$(\Psi_{n,t}, e^{i(2\pi(n+p) + t)x}) = d_p(t) \sum_{n_1} q_{n_1} (\Psi_{n,t}, e^{i(2\pi(n+p-n_1) + t)x}). \quad (45)$$

Now arguing as in the proof of (15), that is, isolating the terms in the right-hand side of (45) containing the multiplicand $(\Psi_{n,t}, e^{i(2\pi n+t)x})$ which occurs in the case $n_1 = p$ and using again (45) for the other terms and etc., we get

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+t)x}) = c_p(t)(\Psi_{n,t}, e^{i(2\pi n+t)x}) + r_m, \quad (46)$$

where

$$r_m = \sum_{n_1, n_2, \dots, n_m} \frac{d_p(t) q_{n_1} q_{n_2} \dots q_{n_m} (q \Psi_{n,t}, e^{i(2\pi(n+p-n_1-n_2-\dots-n_m)+t)x})}{\prod_{s=1,2,\dots,m} [(2\pi n+t)^2 - (2\pi(n+p-n_1-n_2-\dots-n_s)+t)^2]}, \quad (47)$$

$p - n_1 - n_2 - \dots - n_s \neq 0$ for $s = 1, 2, \dots, m$ and $m > p$. The indices n_1, n_2, \dots, n_{p-1} taking part in the expression of $c_p(t)$ satisfy (42). Therefore if $p < 0$, then the set of these indices is empty, that is, $c_p(t) = 0$. Moreover if $p > 0$ then, by (42), the number of summands of $c_p(t)$ is finite.

Now we prove that $r_m \rightarrow 0$ as $m \rightarrow \infty$. Let $M = \sup_n |q_n|$. By (37) $n_k \geq 1$ for $k = 1, 2, \dots, m$ and hence $n_1 + n_2 + \dots + n_s \geq s$. Using this and taking into account that $(q \Psi_{n,t}, e^{i(2\pi(n+p-n_1-n_2-\dots-n_m)+t)x}) \rightarrow 0$ as $m \rightarrow \infty$ we obtain

$$|r_m| \leq |d_p(t)| \prod_{s=1,2,\dots,m} \left(\sum_{j \geq s, j \neq p} \frac{M}{|(2\pi n+t)^2 - (2\pi(n+p-j+t)^2)|} \right) \quad (48)$$

for $m \gg 1$. Clearly there exist $K(t)$ such that

$$\sum_{j \geq s, j \neq p} \left| \frac{M}{[(2\pi n+t)^2 - (2\pi(n+p-j+t)^2)]} \right| \leq K(t). \quad (49)$$

for $s = 1, 2, \dots, m$. Moreover if $s \geq 4(|n| + |p|)$ then

$$\sum_{j \geq s} \left| \frac{M}{[(2\pi n+t)^2 - (2\pi(n+p-j+t)^2)]} \right| < \sum_{j \geq s} \frac{M}{j^2} < \frac{M}{s} \quad (50)$$

Now using (48)-(50) we obtain

$$|r_m| \leq \frac{|d_p(t)| M^{m-4(|n|+|p|)+1} (K(t))^{4(|n|+|p|)-1}}{4(|n| + |p|)(4(|n| + |p|) + 1) \dots m} \quad (51)$$

which implies that $r_m \rightarrow 0$ as $m \rightarrow \infty$. Therefore in (46) letting m tend to infinity we get

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+t)x}) = c_p(t)(\Psi_{n,t}, e^{i(2\pi n+t)x}). \quad (52)$$

This with (43) shows that $(\Psi_{n,t}, e^{i(2\pi n+t)x}) \neq 0$. Therefore, there exists eigenfunction, denoted again by $\Psi_{n,t}$, satisfying (39) and for this eigenfunction, by (52), we have

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+t)x}) = c_p(t).$$

Thus (40) is proved in case (37). The case (38) can be considered in the same way. ■

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